SOME PROBLEMS OF SUMMABILITY OF SPECTRAL EXPANSIONS CONNECTED WITH LAPLACE OPERATOR ON SPHERE.

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Abstract. Solution of some boundary value problems and initial problems in unique ball leads to the convergence and sumability problems of Fourier series of given function by eigenfunctions of Laplace operator on a sphere - spherical harmonics. Such a series are called as Fourier-Laplace series on sphere. There are a number of works devoted investigation of these expansions in different topologies and for the functions from the various functional spaces. In the present work we consider only localization problems in both usual and generalized (almost everywhere localization) senses in the classes of summable functions. We use Chezaro means of the partial sums for study of summability problems. In order to prove the main theorems we obtaine estimations for so called maximal operator estimating it by Hardy-Littlwood's maximal function. Significance of this function is that it majors of many important operators of mathematical physics. For instance, Hardy-Littlewood's maximal function majors Poisson's integral in the space.

1. Introduction

Denote by B^{N+1} a unique ball in R^{N+1} , surface of this ball denote by S^N :

$$S^N = \{x = (x_1, x_2, ..., x_{N+1}) \in R^{N+1} : \sum_{n=1}^{N+1} x_n^2 = 1\}$$

Let x and y arbitrary points in S^N . By $\gamma = \gamma(x, y)$ denote spherical distance between these two points. In fact γ is an angle between vectors x and y . It is clear that $\gamma \le \pi$. By $B(x, r)$ denote a ball on a sphere S^N , with radius r and with the center at a point x :

$$B(x, r) = \{y \in S^N : \gamma(x, y) \le r\}$$

Let Δ_s be Laplace-Beltrami operator on S^N. We have following way to calculate operator Δ_s , using Laplace's operator Δ in R^{N+1} (see for instance in [13].): let $f(x)$ a function determined on S^N ; extend it to R^{N+1}, by putting $\hat{f}(x) = f\left(\frac{x}{|x|}\right)$, $x \in R^{N+1}$. Then $\Delta_s f = \Delta \hat{f}\big|_{S^N}$. Another way of determination of Δ_s is to represent Laplace operator Δ in R^{N+1} by spherical coordinates. In this case it would be easy to "separate" operator Δ_s by separation angled coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_s,$$

2000 *Mathematics Subject Classification.* 35P, 35S, 40xx, 42xx.
Key words and phrases. Spectral expansions, Forier-Laplace series on sphere; summability by Chezaro means; maximal operator and maximal function.

where operator Δ_s can be written in spherical coordinates $(\xi_1, \xi_2, ..., \xi_{N-1}, \zeta)$ as:

$$\Delta_s = \frac{1}{\sin^{N-1}\xi_1}\frac{\partial}{\partial\xi_1}\Big(\sin^{N-1}\xi_1\frac{\partial}{\partial\xi_1}\Big)+\frac{1}{\sin^2\xi_1\sin^{N-2}\xi_2}\frac{\partial}{\partial\xi_2}\Big(\sin^{N-2}\xi_2\frac{\partial}{\partial\xi_2}\Big)+....+$$

$$+\frac{1}{\sin^2\xi_1\sin^2\xi_2\ ...\sin^2\xi_{N-1}}\frac{\partial^2}{\partial\zeta^2}.$$

Operator $-\Delta_s$ as a formal differential operator with domain of definition $C^\infty\big(S^N\big)$ is a symmetric, non negative and its closure $\overline{-\Delta_s}$ is a selfadjoint operator in $L_2\big(S^N\big)$. Eigenfunctions Y^k of the operator $-\Delta_s$, are called spherical harmonics. Spherical harmonics of a degree k and ℓ , $k \neq \ell$ are orthogonal . Corresponding eigenvalues are $\lambda_k = k(k+N-1)$, where $k = 0, 1, 2, ...$, and with frequency a_k equal to the dimension of the space of homogeneous harmonic polynomials of a degree k: $a_k = N_k - N_{k-2}$, where $N_k = \frac{(N+k)!}{N!k!}$. That is why for each k there are a_k number of spherical harmonics $\big\{Y_j^k\big\}\Big|_{j=1}^{a_k}$ corresponding to eigenvalue λ_k . A family of functions $\big\{Y_j^k\big\}\Big|_{j=1}^{a_k}$ is an orthonormal basis in the space of spherical harmonics of a degree k which we denote by $\aleph_k$.

Note that an arbitrary function $f \in L_2\big(S^N\big)$ can be represented in a unique way as Fourier series by spherical harmonics $\big\{Y_j^k\big\}\Big|_{j=1}^{a_k}$. Such a series is called Fourier-Laplace series on sphere:

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{a_k} f_{k,j}Y_j^k(x), \tag{1.1}$$

where $f_{k,j} = \int_{S^N} f(y)Y_j^k(y)d\sigma(y)$, and equality (1.1) should be understanding in sense of $L_2\big(S^N\big)$.

Let denote by $S_n f(x)$ a partial sum of series (1.1). It is clear that in $S_n f(x)$ by changing order of integration and summation one can easily rewrite it as:

$$S_n f(x) = \int_{S^N} f(y)\Theta(x, y, n)d\sigma(y),$$

where a function $\Theta(x, y, n)$ is a spectral function (see in [1]) of a selfadjoint operator $\overline{-\Delta}$ and has a form:

$$\Theta(x, y, n) = \sum_{k=0}^{n}\sum_{j=1}^{a_k} Y_j^k(x)Y_j^k(y), \tag{1.2}$$

and $S_n f(x)$ is called a spectral expansion of an element f correspondin to the operator $\overline{-\Delta}$ (see in [1]).

2. Estimation of maximal operator and almost everywhere convergence.

Determine Chezaro means of order α of partial sums of series (1.1) by equality

$$S_n^\alpha f(x) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha \sum_{j=1}^{a_k} f_{k,j} Y_j^k(x), \tag{2.1}$$

where $A_n^\alpha = \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)m!}$.

Definition 2.1. Series (1.1) is sumable to $f(x)$ by Chezaro means of order α if it is true that

$$\lim_{n\to\infty} S_n^\alpha f(x) = f(x) \tag{2.2}$$

In the this definition equality (2.2) can be understood in any sense (topology). Here in the present article we will consider it in sense of almost every where convergence.

Note that Chezaro means of zero order is coincides with a partial sum $S_n f(x)$ and it is clear that $S_n^\alpha f(x)$ is also can be represented as an integral operator with a kernel which is Chezaro means of the spectral function (1.2)

$$\Theta^\alpha(x,y,n) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha \sum_{j=1}^{a_k} Y_j^k(x) Y_j^k(y), \tag{2.3}$$

Thus formula (2.1) can be written as

$$S_n^\alpha f(x) = \int_{S^N} f(y) \Theta^\alpha(x,y,n) d\sigma(y). \tag{2.4}$$

By $P_k^\nu(t)$ denote Gegenbaur's polinomials (when $\nu = \frac{1}{2}$ Legender's polinomials) [5], [15] . Let's put $\nu = \frac{N-1}{2}$. A function $\Theta^\alpha(x,y,n)$ can be represented in a form (see for instance in [6]) :

$$\Theta^\alpha(x,y,n) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha+1)}{\Gamma(n-k+1)} \big(k+\nu\big)^\nu P_k^\nu(\cos\gamma). \tag{2.5}$$

In investigations of the convergence problems it is important to obtain estimations of so called maximal operator

$$S_*^\alpha f(x) = \sup_{n>1} |S_n f(x)|. \tag{2.6}$$

For any locally sumable on S^N function $f(x)$ by $f^*(x)$ denote Hardy-Littlwood's maximal function determined as

$$f_*(x) = \sup_{r>0} \frac{1}{mesB(x,r)} \int_{B(x,r)} |f(y)| d\sigma(y). \tag{2.7}$$

In this section we prove that this function majors operator determined by (2.6). First we note here some well known properties of Hardy-Littlwood's maximal function.

There exists a constant $c = c(N)$, depending only on dimension of sphere such that for all $f \in L_1\big(S^N\big)$ and $F_\mu = \{x \in S^N : f^*(x) > \mu > 0\}$ following estimation

is valid

$$mesF_\mu \leq \frac{c \parallel f \parallel_1}{\mu} \tag{2.8}$$

where $\parallel f \parallel_1$ denotes a norm of f in $L_1\big(S^N\big)$.

Moreover, there is a constant $b = b(p, N)$, that depends only on dimension N and on an index of summability $p > 1$ such that

$$\|f^*\|_p \leq b\|f\|_p. \tag{2.9}$$

for all $f \in L_p\big(S^N\big)$. Estimation (2.8) is well known as Hardy-Littlwood's function has weak type of (1,1) (means it is weak bounded from $L_1\big(S^N\big)$ to $L_1\big(S^N\big)$) and (2.9) says that it has strong (p, p) type when $p > 1$.

Let's denote by $\overline{x}$ a point that diametrically opposite to a point x of the sphere: $\gamma(x, \overline{x}) = \pi$. In the present section we will prove following estimation for maximal operator (2.6):

Theorem 2.2. *Let $f(x)$ a summable on a sphere function and let $\alpha > \frac{N-1}{2}$. Then there is a constant c_α not depending on f such that*

$$S_*^\alpha f(x) \leq c_\alpha\big(f^*(x) + f^*(\overline{x})\big), \tag{2.10}$$

where f^* is Hardy-Littlewood's maximal function.

Proof. For the kernel (2.5) following estimations are valid [6]:

if $\alpha > -1$ and $|\frac{\pi}{2} - \gamma| \leq \frac{n}{n+1}\frac{\pi}{2}$, then for $n \to \infty$

$$\Theta^\alpha(x, y, n) = \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \frac{\Gamma\Big(n + \frac{N+1}{2}\Big)}{\Gamma\Big(\frac{N+1}{2}\Big)} \times \frac{\sin\Big(\Big(n + \frac{N+1}{2}\Big)\gamma - \Big(\frac{N-1}{2} + \frac{\alpha}{2}\Big)\frac{\pi}{2}\Big)}{\Big(2\sin\gamma\Big)^{\frac{N-1}{2}}\Big(2\sin\frac{\gamma}{2}\Big)^{1+\alpha}} +$$

$$+ \frac{O\Big(n^{\frac{N-1}{2}-\alpha-1}\Big)}{\Big(\sin\gamma\Big)^{\frac{N+1}{2}}\Big(\sin\frac{\gamma}{2}\Big)^{1+\alpha}} + \frac{O\Big(\frac{1}{n}\Big)}{\Big(\sin\frac{\gamma}{2}\Big)^{1+N}} \tag{2.11}$$

if $\alpha > -1$ and $0 < \gamma_0 \leq \gamma \leq \pi$, then for $n > 1$

$$|\Theta^\alpha(x, y, n)| \leq c\ n^{N-1-\alpha} \tag{2.12}$$

if $\alpha > -1$ and $0 \leq \gamma \leq \pi$, then for $n > 1$

$$|\Theta^\alpha(x, y, n)| \leq c\ n^N \tag{2.13}$$

We will use these asymptotic formulas for estimation of Chezaro means (2.4) . For that we transform integral in the right side of (2.4) to the following form:

$$\int_{\gamma \leq \frac{1}{n}} + \int_{\frac{1}{n} \leq \gamma \leq \frac{\pi}{2}} + \int_{\frac{\pi}{2} \leq \gamma \leq \pi - \frac{1}{n}} + \int_{\frac{1}{n} \leq \gamma \leq \pi}$$

Then using correspondingly formulas (2.11)-(2.13), obtain

$$|S_n^\alpha f(x)| \ \leq \ c_1\, n^N \int_{\gamma \leq \frac{1}{n}} |f(y)| d\sigma(y) + c_2\, n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n} \leq \gamma \leq \frac{\pi}{2}} \frac{|f(y)|}{\Big(\sin\gamma\Big)^{\frac{N+1}{2}+\alpha}} d\sigma(y) +$$

$$+ c_3\, n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n}\le\gamma\le\frac{\pi}{2}} \frac{|f(y)|}{\Big(\sin\gamma\Big)^{\frac{N+3}{2}+\alpha}} d\sigma(y) + c_4\, n^{-1} \int_{\frac{1}{n}\le\gamma\le\frac{\pi}{2}} \frac{|f(y)|}{\Big(\sin\gamma\Big)^{N+1}} d\sigma(y) +$$

$$+ c_5\, n^{\frac{N-1}{2}-\alpha} \int_{\frac{\pi}{2}\le\gamma\le\pi-\frac{1}{n}} \frac{|f(y)|}{\Big(\sin\gamma\Big)^{\frac{N-1}{2}+\alpha}} d\sigma(y) + c_6\, n^{\frac{N-3}{2}-\alpha} \int_{\frac{\pi}{2}\le\gamma\le\pi-\frac{1}{n}} \frac{|f(y)|}{\Big(\sin\gamma\Big)^{\frac{N+1}{2}+\alpha}} d\sigma(y) +$$

$$+ c_7\, n^{-1} \int_{\frac{\pi}{2}\le\gamma\le\pi-\frac{1}{n}} |f(y)| d\sigma(y) \; + \; c_8\, n^N \int_{\frac{1}{n}\le\gamma\le\pi} |f(y)| d\sigma(y) \tag{2.14}$$

Further we continue to estimate through Hardy-Littlwood's maximal function.

$$|S_n^\alpha f(x)| \;\le\; c_0 \Big\{ \, n^N \int_{\gamma\le\frac{1}{n}} |f(y)| d\sigma(y) +$$

$$+ n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \frac{1}{\Big(\sin\gamma\Big)^{\frac{N+3}{2}+\alpha}} \, d \int_{\gamma<r} |f(y)| \, d\sigma(y) +$$

$$+ n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \frac{1}{\Big(\sin\gamma\Big)^{\frac{N+3}{2}+\alpha}} \, d \int_{\gamma<r} |f(y)| \, d\sigma(y) +$$

$$+ \frac{1}{n} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \frac{1}{\Big(\sin\gamma\Big)^{N+1}} \, d \int_{\gamma<r} |f(y)| \, d\sigma(y) +$$

$$+ n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \frac{1}{\Big(\sin\gamma\Big)^{\frac{N-1}{2}}} \, d \int_{\gamma<r} |f(y)| \, d\sigma(y) +$$

$$+ n^{\frac{N-3}{2}-\alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \frac{1}{\Big(\sin\gamma\Big)^{\frac{N+1}{2}}} \, d \int_{\gamma<r} |f(y)| \, d\sigma(y) +$$

$$+ \frac{1}{n} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \, d \int_{\gamma<r} |f(y)| \, d\sigma(y) \Big\} \tag{2.15}$$

By replacing with maximal function we obtain

$$|S_n^\alpha f(x)| \;\le\; c_0 \Big\{ \Big(1 + \; n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} r^{\frac{N-3}{2}-\alpha} dr \; + \; \frac{1}{n} \int_{\frac{1}{n}}^{\frac{\pi}{2}} \frac{dr}{r^2}\Big) \, f^*(x) +$$

$$+ \Big(1 + \; n^{\frac{N-1}{2}-\alpha} \int_{\frac{1}{n}}^{\frac{\pi}{2}} r^{\frac{N-1}{2}} dr \; + \; \frac{1}{n} \int_{\frac{1}{n}}^{\frac{\pi}{2}} dr^N\Big) \, f^*(\overline{x}) \;\Big\}$$

Taking into consideration that $\alpha > \frac{N-1}{2}$, one can obtain that all integrals in last inequality finite. Then using definition of maximal function we obtain estimation (2.10). Theorem 2.2 is proved $\square$

Note that theorem 2.2. specified obtained earlier estimation of maximal operator.[1] From the theorem 2.2 it is easy to obtain following corollaries.

[1]Estimation (2.10) specifies a similar estimation obtained by A. Bonami , J. Clerc [4] and such a specification was required due to necessity to estimate Chezaro means of critical exponent (see in section 3 below). For the first time estimation (2.10) was obtained by the author of the present paper in [8] (see also in [9])

Corollary 2.3. *Let $f \in L_1\big(S^N\big)$. If $\alpha > \frac{N-1}{2}$, then for maximal operator (2.6) following estimation if valid*

$$mes\big\{x \in \big(S^N\big) : S_*^\alpha f(x) > \mu > 0\big\} \ \leq \ \frac{c\ \|f\|_1}{\mu} \tag{2.16}$$

Proof. Inequality (2.16) immediately follows from estimation (2.10) and inequality (2.8). Corollary is proved. □

Corollary 2.4. *Let $f \in L_1\big(S^N\big)$ and $\alpha > \frac{N-1}{2}$. Then almost everywhere on sphere*

$$\lim_{n\to\infty} S_n^\alpha f(x) = f(x). \tag{2.17}$$

Proof. Equality (2.17) follows from estimation (2.16) . Corollary is proved. □

Using interpolation theorem of E. Stein [14] and estimation for Hardy-Littlwood maximal function (2.9) from the theorem 2.2 we obtain (see: [4])

Corollary 2.5. *Let $f \in L_p\big(S^N\big),\ p > 1$ and $\alpha > (N-1)\big(\frac{1}{p} - \frac{1}{2}\big)$. Then*

(i) *For maximal operator (2.6) following inequality is true*

$$\|S_*^\alpha f\|_p \ \leq \ c\|f\|_p \tag{2.18}$$

(ii) *Almost everywhere on sphere we have following equality*

$$\lim_{n\to\infty} S_n^\alpha f(x) = f(x). \tag{2.19}$$

3. Generalized localization of Chezaro means of order $\frac{N-1}{2}$ of Fourier-Laplace series of functions from L_1 .

Definition 3.1. *Let V a subdomain of a sphere, it can also coincides with S^N. We say that for Chezaro means $S_n^\alpha f(x)$ it is true generalized principles of localization in a class of functions $L_p\big(S^N\big)$, if for an arbitrary function f form $L_p\big(S^N\big)$ such that $f(x) = 0$ when $x \in V$, almost everywhere on V*

$$\lim_{n\to\infty} S_n^\alpha f(x) = 0.$$

Main result of the present section is a following theorem.

Theorem 3.2. *Let $f \in L_1\big(S^N\big)$ and let $f(x) = 0$, when $x \in V \subset S^N$. Then almost everywhere on V Fourier-Laplace series of a function f convergence to zero by Chezaro means of the order $\alpha = \frac{N-1}{2}$.*

From theorem 3.2. it follows that principles of generalized localization for the Fourier-Laplace series on a sphere is valid in critical index $\alpha = \frac{N-1}{2}$ of Chezaro means in a class of functions L_1. In order to prove the theorem we should estimate maximal operator (2.6) with critical exponent.[2]

[2]Exponent $\frac{N-1}{2}$ was termed by Bochner [3] as the critical exponent. It was, in particular, justified by the fact that localization principal holds for spherical partial Fourier integrals with the critical (and above) exponent, and but not for exponent below it. However, for Fourier multiple series (as well as for Fourier-Laplace series on sphere) with spherical sums localization principle (in usual sense) is not correct in critical exponent [14]. Localization of the critical exponent for the Chezaro means of spherical expansions of distributions are studied in [10] - [12]. Meantime a localization of the partial sums (means of zero exponent) in class L_2 is another critical case (Luzin's problem, see [1]). This problem was studied by Bastis I.J. in [2] and Meaney C. in [7].

Lemma 3.3. *Let $f(x)$ a summable function on a sphere and let denotes $\overline{x}$ diametrically opposite point to x and let $\alpha \geq \frac{N-1}{2}$. If $f(x) = 0$, when $x \in V \subset S^N$, then there exists a constant c_α not depending on f such that at any point of the domain $V_1 \subset V$, where $\gamma(V_1, \partial V) > 0$, following inequality is valid*

$$S_*^\alpha f(x) \leq c_\alpha \ f^*(\overline{x}). \tag{3.1}$$

here f^* is Hardy-Littlwood's maximal function, ∂V is a spherical boundary of the domain V .

Proof. We will choose a constant r_0, $(r_0 > 0)$, such that for an arbitrary x from V_1 spherical ball of a radius equal to r_0 with the center at x is a subset of V . For estimation of $S_*^\alpha f(x)$ we transform integral in right side of (2.4) as

$$\int_{\gamma \leq r_0} + \int_{r_0 \leq \gamma \leq \pi - \frac{1}{n}} + \int_{\pi - \frac{1}{n} \leq \gamma \leq \pi} .$$

Note that because of $f(x) = 0$ on V_1 , first integral is equal to zero. Thus from (2.11)-(2.12) we obtain

$$|S_n^\alpha f(x)| \ \leq \ c \Big\{ \ n^{\frac{N-1}{2} - \alpha} \int_{r_0 \leq \gamma \leq \pi - \frac{1}{n}} \frac{|f(y)|}{\Big(\sin \gamma\Big)^{\frac{N-1}{2}}} d\sigma(y) +$$
$$+ \ n^{\frac{N-3}{2} - \alpha} \int_{r_0 \leq \gamma \leq \pi - \frac{1}{n}} \frac{|f(y)|}{\Big(\sin \gamma\Big)^{\frac{N+1}{2}}} d\sigma(y) +$$
$$+ \ n^{-1} \int_{r_0 \leq \gamma \leq \pi - \frac{1}{n}} |f(y)| d\sigma(y) + \ n^{N-1-\alpha} \int_{\frac{1}{n} \leq \gamma \leq \pi} |f(y)| d\sigma(y) \Big\} \tag{3.2}$$

where a constant c depends only on r_0 . Then each term in right side of (3.2) estimate by Hardy-Littlwood's maximal function

$$|S_n^\alpha f(x)| \ \leq \ c \Big\{ \ n^{\frac{N-1}{2} - \alpha} \int_{r_0 \leq \gamma \leq \pi - \frac{1}{n}} \frac{|f(y)|}{\Big(\sin \gamma\Big)^{\frac{N-1}{2}}} d\sigma(y) +$$
$$+ \ n^{\frac{N-3}{2} - \alpha} \int_{r_0 \leq \gamma \leq \pi - \frac{1}{n}} \frac{|f(y)|}{\Big(\sin \gamma\Big)^{\frac{N+1}{2}}} d\sigma(y) +$$
$$+ \ n^{-1} \int_{r_0 \leq \gamma \leq \pi - \frac{1}{n}} |f(y)| d\sigma(y) + \ n^{N-1-\alpha} \int_{\frac{1}{n} \leq \gamma \leq \pi} |f(y)| d\sigma(y) \Big\} \tag{3.3}$$

Here a symbol $\overline{\gamma}$ is a spherical distance between points $\overline{x}$ and y , $\overline{\gamma} = \gamma(\overline{x}, y)$. Replacing with the maximal function we obtain

$$|S_n^\alpha f(x)| \ \leq \ c_0 \Big\{ \ 1 + \int_{\frac{1}{n}}^{\pi - r_0} \Big(\sin r\Big)^{-\frac{N-1}{2}} r^{N-1} dr \Big\} f^*(\overline{x}) \tag{3.4}$$

and a constant c_0 in (3.4) does not have singularities when $\alpha = \frac{N-1}{2}$. Lemma 3.3 is proved. □

Lemma 3.4. *Let* $f \in L_1(S^N)$ *and this function satisfies the conditions of lemma 3.3 . If* $\alpha = \frac{N-1}{2}$ *, then for maximal operator (2.6) following estimation is valid*

$$mes\Big\{x \in V_1 : S_*^\alpha f(x) \ > \ \mu \ > \ 0 \ \Big\} \ \le \ c\frac{\| f \|_1}{\mu} \tag{3.5}$$

Proof. Inequality (3.5) immediately follows from lemma 3.3 and from the fact that Hardy-Littlwood's maximal function is weak bounded from $L_1(S^N)$ to $L_1(S^N)$. Lemma 3.4 is proved. □

Statements of theorem 3.2 immediate follows from estimation (3.5).

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